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Turbulence Spectra

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The scaling invariance of the Navier–Stokes equations in the limit of infinite Reynolds number is used to derive laws for the inertial range of the turbulence spectrum. Whether the flow is homogeneous or not, the spectrum is chosen to be that given by a well-chosen biorthogonal decomposition. If the flow is homogeneous, this spectrum coincides with the classical Fourier (energy) spectrum which exhibits Kolmogorov's $k^{-5/3}$ power law if the scaling exponent is assumed to be 1/3. In the more general case where the homogeneity assumption is relaxed, the spectrum is discrete and decays exponentially fast under the assumption that the flow is invariant (in a deterministic or statistical sense) under only one subgroup of the scaling coefficient λ of one scaling group of the equations (corresponding to one value of the scaling exponent). If the flow is invariant under two subgroups of scaling coefficients λ and λ', the spectrum becomes maximal, equal to R_+. Finally, when a full symmetry, namely an invariance under a whole group, is assumed and the spectrum becomes continuous, the decaying law for the spectral density is derived and found to be independent of the specific value of h. These ideas are then applied to locally self-similar flows with multiple dilation centers (localized in space and time) and multiple scaling exponents, extending the concept of multifractals to space and time.

KEY WORDS: Turbulence; biorthogonal decomposition; self-similarity; fractals; multifractals; wavelets.

INTRODUCTION

It is well known that a theory for fully developed turbulence is far from being complete. The hope is that, in the limit of very high (eventually

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infinite) Reynolds number, some simple scaling laws are valid so that a *universal* (or scaling) theory can be derived. Although Kolmogorov's $k^{-5/3}$ universal law (K41)[16] for the spatial Fourier spectrum of homogeneous turbulence was derived 50 years ago, not much progress has been accomplished since then regarding the derivation of an analogous scaling law of fully developed inhomogeneous turbulence. This work is an attempt in this direction.

Following Frisch's "modern" viewpoint,[12] we now briefly review the basis of Kolmogorov's scaling theory. The Navier–Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\,\nabla^2 \mathbf{u}$$
$$\nabla \cdot \mathbf{u} = 0 \qquad (*)$$

where $\mathbf{u}$ denotes the velocity field, p the pressure, and ν the kinematic viscosity, are invariant under various symmetries. The K41 hypothesis are equivalent to assuming that the (asymptotic) symmetries of the Navier–Stokes equations are valid in a statistical sense, in particular the (statistical) translation and rotation symmetries (i.e., homogeneity and isotropy) and the scaling invariance.[12] Here, the meaning of "statistics" is in the time average sense.

In this paper, we are interested in relaxing the hypotheses of homogeneity and isotropy of the K41 theory to derive a scaling law for the spectrum of fully developed inhomogeneous turbulence. Since in this case it is not possible to use the Fourier spectrum, we propose to use the spectrum of the operator U (as defined below) introduced in the derivation of the biorthogonal decomposition.[3] We reserve here the terminology "biorthogonal decomposition" for any decomposition which consists in an expansion of a space-time function—or signal—$u(x, t)$ (here the velocity) into spatial orthogonal modes in a Hilbert space $H(X)$ $(x \in X)$ and temporal orthogonal modes in a Hilbert space $H(T)$ $(t \in T)$ and which defines a unique dispersion relation between both sets of modes. The operator U is defined as follows:

$$U\colon\quad H(X) \to H(T)$$

such that

$$\forall \varphi \in H(X), \qquad (U\varphi)(t) = \int_X u(x, t)\, \varphi(x)\, d\mu(x) \qquad (**)$$

where $d\mu(x)$ [resp. $d\tilde{\mu}(t)$] denotes the measure defining the scalar product in $H(X)$ [resp. $H(T)$]. The biorthogonal decomposition of u is the spectral analysis of the operator U. Two particular examples of such a decomposition are the two-dimensional Fourier decomposition for plane waves and the probability theory tool called the proper orthogonal decomposition, Karhunen–Loève expansion, or principal component analysis viewed from a deterministic approach [which uses $L^2(X)$] when the statistical average needed here is chosen to be the time average. The application of the latter to turbulence was first proposed and developed by Lumley,[19,20] discussed more recently in a number of works (e.g., Sirovich[24]), and applied to a rather large number of flows (see, e.g., Aubry *et al.*,[2] Berkooz *et al.*,[5] Chambers *et al.*,[8] Glauser *et al.*,[13] Glezer *et al.*,[14] Deane *et al.*,[9] Deane and Sirovich,[10] Moin and Moser,[21] Sirovich *et al.*[25]). In ref. 3, the introduction of two Hilbert spaces as well as the generalization to other Hilbert spaces than $L^2(X)$ and $L^2(T)$ are of primary importance, as we show in this paper. This definition of the turbulence spectrum as that of the spectrum of the operator U for an inhomogeneous flow is natural since each eigenvalue represents the contribution to the square root of the kinetic energy of the flow (in the sense of the two Hilbert spaces defined above) of the associated eigenfunctions or spatiotemporal structures. It has been used in the past in the proper orthogonal decomposition context (see, e.g., refs. 11 and 15).

However, in contrast with the previously mentioned studies, we cannot merely refer to the decompositions of the correlation functions only. While the latter correspond to the operators U^*U and UU^*, we are interested in the operator U itself. This leads us to compare the scaling invariance in a statistical sense with the symmetry of U.[4] In particular, we investigate the implications of the former on the spectrum of U. Therefore, in order to study the signal u by means of a self-adjoint operator, we consider the operator

$$\begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix}$$

defined on $H(X) \oplus H(T)$.

The paper is organized as follows. In Section 1, we define the operator U in a more general context than in our earlier work[3,4] to account for cases where the spectrum is continuous or where U is an unbounded operator with eventually a singular kernel u. This leads to a specific (generalized) form of the biorthogonal decomposition derived in Appendix A, which will be needed in the subsequent sections. In Section 2, assuming the presence of scaling symmetries with one dilation factor and one scaling exponent in the inertial range of a fully developed turbulent flow, we derive the exponentially decreasing spectrum in this range. We then show in Section 3 that this scaling law, valid whether the flow is homogeneous or not, gives the celebrated $k^{-5/3}$ power law when the flow is assumed to be homogeneous. In Section 4, we investigate the spectrum of a flow which is invariant under two (or more) representations of the same scaling group (i.e., with the same scaling exponent). Assuming that the scaling symmetry is satisfied with only one or several dilation factors is equivalent to supposing lacunarity in the flow (by analogy with lacunarity in fractals). In the case of no lacunarity, the spectrum is R_+ and if it is assumed to be continuous, a decaying law for the spectral density is derived. Finally, in Section 5, we extend the previous results of global self-similarity for which the scaling exponent is a function of the dilation center location in both space and time. This permits the treatment of eventual singularities which may occur in the space/time domain, following the multifractality concept of Parisi and Frisch[22] (extended to spatiotemporal systems). In Appendix A, we generalize the biorthogonal decomposition in order to cover the case of a continuous spectrum as well as the case of singular kernels. In Appendix B, we point out a connection between our results and the wavelet techniques of signal analysis.

1. THE OPERATOR U

In this work, we consider a spatiotemporal signal u which, in the case of the Navier–Stokes equations, represents the velocity field, which we denote by $\mathbf{u}(\mathbf{x}, t)$ with $\mathbf{x} \in R^3$, $t \in R$. Since what follows is valid whether we consider the full velocity field or only one component, for the sake of simplicity, we will consider one component $u(\mathbf{x}, t)$ only (see ref. 3 for the form that takes the biorthogonal decomposition if $\mathbf{u}$ is a vector). Here, it is possible that the operator U defined as[3]

$$U\colon \quad L^2(R^3, d\mathbf{x}) \to L^2(R, dt)$$

such that

$$(U\varphi)(t) = \int u(\mathbf{x}, t)\, \varphi(\mathbf{x})\, d\mathbf{x} \tag{1.1}$$

is neither compact nor bounded. We thus naturally consider the operator U as an integral operator in various Hilbert spaces, a possibility which was partially examined in refs. 3 and 4. In this regard, we consider two cases interesting from both the physical and mathematical viewpoints.

In the first case, the spatial domain is infinite in R^3 as well as the temporal domain in R. This corresponds to the asymptotic dynamics of an open or spatially periodic flow. We can then use the definitions introduced in ref. 3 and restated in (1.1), where the domain of U is

$$D(U) = \{\varphi \in L^2(R^3, d\mathbf{x}),\ U\varphi \in L^2(R, dt)\} \tag{1.2}$$

so that U^*U and UU^* are self-adjoint, positive operators. This definition, however, is not always possible since, for certain kernels $u(\mathbf{x}, t)$, $D(U)$ defined as in (1.2) may not be dense in $L^2(R^3, dx)$, mainly due to local singularities or pathological asymptotic properties; $u(\mathbf{x}, t)$ can then be a Carleman or generalized kernel, which requires a special treatment presented in Appendix A. In brief, the method exposed there uses the Hilbert spaces $L^2(R^3, d\mu(\mathbf{x}))$ and $L^2(R, d\tilde{\mu}(t))$, where the measures $d\mu$ and $d\tilde{\mu}$ are chosen so that the corresponding spaces contain the generalized eigenfunctions of U (see below for the exact meaning of "generalized"). Then the kernel $u(\mathbf{x}, t)$ may be defined only in a weak sense by

$$(Uf, \tilde{g}) = \iint u(\mathbf{x}, t) f(\mathbf{x}) \tilde{g}(t) \, d\mathbf{x} \, dt \tag{1.3}$$

where f and $\tilde{g}$ belong to some subspaces of the Hilbert spaces $L^2(R^3, d\mu(\mathbf{x}))$ and $L^2(R, d\tilde{\mu}(t))$. These subspaces are analogous to the space of test functions in the theory of distributions.

In the second case, the theory is local in both space and time. We then arbitrarily choose a compact $K \subset R^3$—a cube, for instance—and a finite interval $T \subset R$ and we define the scalar products by integrating over these domains and normalizing:

$$(\varphi_1, \varphi_2)_K = \frac{1}{|K|} \int_K \varphi_1(\mathbf{x}) \overline{\varphi_2(\mathbf{x})} \, d\mathbf{x} \tag{1.4}$$

and

$$(\psi_1, \psi_2)_T = \frac{1}{|T|} \int_T \psi_1(t) \overline{\psi_2(t)} \, dt \tag{1.5}$$

We call the Hilbert spaces thus defined $H(K)$ and $H(T)$. Of course, these spaces depend on the "windows" K and T and we will have to compare these different "realizations." Even if only one function $u(\mathbf{x}, t)$ is considered, it is clear that the choice of a pair of such Hilbert spaces $(H(K), H(T))$, defined on specific integration domains K and T as in (1.4) and (1.5), leads to different integral operators U in each case. It follows that the spectral properties of U, which depend sensitively on the chosen representation, can change accordingly. Moreover, it is not always possible to consider the infinite limits $K \to R^3$ and $T \to R$ and, even if the latter exist, we may not recover the properties of U as defined in (1.1). In this paper, we are interested in the scaling properties of the spectrum of U in the presence of symmetries. Although the decay rates of the spectra corresponding to different representations may be different, they are linked to each other and their ratio can be easily calculated (see Appendix A). It is also worth noticing that the ratio of two decay rates corresponding to the presence of two different symmetries is independent of the selected representation.

Additional knowledge of the kernel $u(\mathbf{x}, t)$ would be necessary to characterize the spectral measure of U. However, the fact that the scaling properties of the spectrum as well as the form of the generalized biorthogonal decomposition can be deduced from its symmetries seems remarkable. The latter can be written as

$$u(\mathbf{x}, t) = \int_0^{+\infty} A \sum_{n=1}^{N_A} \varphi_n^A(\mathbf{x})\, \psi_n^A(t)\, dm(A) \tag{1.6}$$

where the generalized topos φ_n^A and chronos ψ_n^A are to be taken in the spaces mentioned above, $L^2(R^3, d\mu(\mathbf{x}))$ and $L^2(R, d\tilde{\mu}(t))$ [see Appendix A for the exact meaning of (1.6)]. Notice that we can recover a discrete spectrum and a compact operator as treated in refs. 3 and 4 (except when the spectrum is degenerate with an infinite degeneracy) by considering the operator $\tilde{P}_\Delta U P_\Delta$, where $\tilde{P}_\Delta$ and P_Δ are the temporal and spatial spectral operators corresponding to a bounded interval of the spectrum in R_+. This is also the way to proceed to restrict ourselves to the inertial range and eliminate both ends of the spectrum, namely the energy-containing and dissipative ranges (which is a necessary step in the following sections). It is also worth pointing out that our method does not require the elimination of the singularity at $0 \in \Delta$, as shown in Appendix A.

Finally, in order to avoid any confusion with the Fourier spectrum extensively used in hydrodynamics and other fields, we refer to the spectrum of the operator U as the "kinetic spectrum" (we avoid the terminology "energy spectrum" used in turbulence for the Fourier spectrum).

2. SCALING AND SYMMETRIES

We know that the Navier–Stokes equations, in the limit of zero viscosity, are invariant under the scaling groups of transformations:

$$\begin{aligned} \mathbf{x} &\to \lambda \mathbf{x} \\ t &\to \lambda^{1-h} t \\ u &\to \lambda^h u \end{aligned} \tag{2.0}$$

and that the fully developed turbulent solutions themselves may be invariant, in a "statistical" sense and in the inertial subrange of the spectrum, under one or several subgroups of one of these groups (Kolmogorov's hypothesis[12]). We will examine the statistical aspect of the problem at the end of this section. First, we assume that the turbulent field itself satisfies the scaling invariance:

$$u(\mathbf{x}, \lambda^{1-h}t) = \lambda^h u(\lambda^{-1}\mathbf{x}, t), \qquad \mathbf{x} \in R^3, \quad t \in R \tag{2.1}$$

Although the Navier–Stokes equations are invariant under the transformations (2.0) for all h and all $\lambda > 0$, we suppose for the moment that (2.1) is true only for a certain $\lambda > 0$ and for a certain h. Remark that if (2.1) is valid for a coefficient λ, then it is also valid for λ^{-1}. Therefore, in what follows, we can take $\lambda > 1$ or $\lambda < 1$, the choice being only a matter of convention. We take $\lambda < 1$, since then the eigenvalues of U are ordered in a decreasing order [see (2.5) and (4.5) below].

Following the technique developed in ref. 4 and recalled in the Introduction, we define two quasisymmetries, namely two representations of the multiplicative group R_+ on the two Hilbert spaces used in the definition of U. First, putting aside the general case of Hilbert spaces defined with weights—treated in Appendix A—we now consider $L^2(R^3, d\mathbf{x})$ and $L^2(R, dt)$. The first representation is defined on $L^2(R^3)$ by

$$(S_\lambda \varphi)(\mathbf{x}) = \lambda^{-3/2}\varphi(\lambda^{-1}\mathbf{x}) \tag{2.2}$$

and the second one on $L^2(R)$ by

$$(\tilde{S}_\lambda \psi)(t) = \alpha^{-1/2}\psi(\alpha^{-1}t) \tag{2.3}$$

where $\alpha = \lambda^{1-h}$. In the definitions (2.2) and (2.3), $\lambda \in R^+$ is the group element and $h \in R$ is fixed. We then immediately obtain the commutation relations

$$\tilde{S}_\lambda U = \tilde{\beta} U S_\lambda \tag{2.4}$$

with $\tilde{\beta} = \lambda^{-(h/2+2)}$. When the operator U is not bounded, these expressions should be understood in the sense of Appendix A. When the kernel is singular, the analysis needs to be carried out with Hilbert spaces defined with weights (see Appendix A), which only changes the exponent of λ in $\tilde{\beta}$. Iterating (2.4), we obtain the same expressions when λ is replaced by λ^n, $n \in Z$, which leads to a scaling exponential law of the spectrum.[4] If the symmetry (2.4) is not satisfied with another $\lambda' \in R_+$, $\lambda' \neq \lambda$, then Theorem 4.1 of ref. 4 [with $\mathrm{Ker}(U) = \mathrm{Ker}(S_\lambda)$], which can be easily extended to the results of Appendix A, shows that the spectrum is discrete and consists of

$$A_n = \tilde{\beta}^{-n} A_0, \qquad n \in Z \tag{2.5}$$

Then, the topos (resp. chronos) are all images of the basic topo φ_0 (resp. the basic chrono ψ_0) by the action of S_λ (resp. $\tilde{S}_\lambda$):

$$\varphi_n = S_{\lambda^n} \varphi_0 \tag{2.6}$$

$$\psi_n = \tilde{S}_{\lambda^n} \psi_0 \tag{2.7}$$

The analogy with the wavelet transforms is pointed out in Appendix B.

If the scaling invariance (2.1) is satisfied for all $\lambda \in R_+$ (with a fixed h), then the spectrum of U is equal to R_+. Reiterating the same analysis in the Hilbert spaces $H(K)$ and $H(T)$, with the scalar products defined in (1.4) and (1.5), we easily check that, in this case, the decay law (2.5) is given by

$$\tilde{\beta} = \lambda^{-h} \tag{2.8}$$

This decay rate is independent of the window $K \times T$ as long as the scaling exponent h and the scaling coefficient λ themselves do not depend on $K \times T$. We will come back to this important issue in Sections 4 and 5. We should mention here that the scaling relation (2.4) is not, in the strict sense, equivalent to a fractal geometry of the velocity u since it involves a spatial as well as a temporal scaling. The fractal geometry, nevertheless, is recovered for the spatial and temporal two-point correlation functions, i.e., the kernels $R(\mathbf{x}, \mathbf{y})$ and $\tilde{R}(t, s)$ of the operators U^*U and UU^*,[1,3] since (2.4) implies that these operators enjoy the symmetry properties

$$S_\lambda^{-1} U^* U S_\lambda = \tilde{\beta}^{-2} U^* U \tag{2.9}$$

and

$$\tilde{S}_\lambda^{-1} U U^* \tilde{S}_\lambda = \tilde{\beta}^{-2} U U^* \tag{2.10}$$

We can now question what remains from the previous analysis when the scaling property (2.1) is valid only in a statistical sense. If the ensemble average is replaced by a temporal average (as is often indeed the case in turbulence), then the second-order statistical correlations are simply described by the kernels of the operators U^*U and UU^*. We now have two possibilities as follows:

(i) In the first one, we assume that the statistical scaling invariance is expressed by the simultaneous realization of (2.9) and (2.10), namely the two-point spatial correlation operator commutes with S_λ (up to the factor $\tilde{\beta}^2$) and the temporal two-point correlation operator commutes with $\tilde{S}_\lambda$ (up to the factor $\tilde{\beta}^2$). The question is now whether (2.9) and (2.10) imply (2.4). This leads to the following issue: can one reconstruct U from UU^* and U^*U? The answer to the latter is well known and can be deduced from the polar decomposition of the operator U. Given the positive part of an operator and that of its adjoint, the operator U is unique up to a phase function on the spectrum if the latter is nondegenerate, namely $N_A = 1$ for m-almost all A in (1.6). In our case, we have fixed the phase function by requiring a positive spectrum. In the other case, i.e., when the spectrum of U (and thus those of UU^* and U^*U) is degenerate, then U is fixed up to arbitrary rotations inside each degenerate eigenspace. However, this ambiguity does not modify the symmetry properties, in the sense that for two different choices U and U', the corresponding symmetries are equivalent, the intertwining operators being precisely the rotations which connect U and U'. An interesting point is that the spectrum is not affected by this indeterminacy in U and consequently, it is identical whether the symmetry is statistical [Eqs. (2.9), (2.10)] or instantaneous [Eq. (2.4)].

(ii) As in ref. 1, it should be mentioned that the kernel $R(\mathbf{x}, \mathbf{y})$ is of common use in turbulence, while the kernel $\tilde{R}(t, s)$ is not. Also, the symmetry satisfied by $R(\mathbf{x}, \mathbf{y})$—often assumed in turbulence theory, e.g., Kolmogorov[16]—is supported by experimental results (to the extent recalled in the Introduction). On the contrary, since, to our knowledge, the temporal two-point correlation function has not been experimentally or numerically investigated (nor has the instantaneous three-dimensional velocity field itself), it is difficult to see whether (2.10) is valid or not. Consequently, we now relax (2.10) and assume that the "statistical" symmetry consists in (2.9) only. Then it is easy to check that (2.5) still holds, with the same decay rate $\tilde{\beta}$, which is consistent with the fact that the square of the spectrum of U is the same as that of U^*U. While (2.6) is still satisfied, (2.7) and (2.10) are now valid with a symmetry $\tilde{S}$ which may be different from $\tilde{S}_\lambda$. The question is then the following: Can one reconstruct the symmetry $\tilde{S}$ satisfied by UU^* from the symmetry S_λ satisfied by U^*U? Since the operator U realizes an isomorphism between $\chi(X)$ and $\chi(T)$ and the operator S_λ preserves $\chi(X)$, $\tilde{S}$ defined as

$$\forall \psi_n^A \in \chi(T), \qquad \tilde{S}\psi_n^A = \frac{1}{A} U S_\lambda \varphi_n^A \tag{2.11}$$

and $\tilde{S} = 0$ on the orthogonal of $\chi(T)$, for m-almost all A and $n \leqslant N_A$ in the cases of Appendix A, is a representation of R_+ in $\chi(T)$ which realizes the desired symmetry. Moreover, if one supposes that this symmetry is implemented by an invertible and differentiable transformation acting on the variable t, then a dilation on time is obtained.[4] We are now back to (i). However, it would be interesting to investigate the nature of $\tilde{S}$ from experimental or numerical data. While this would not have any effect on the turbulence spectra, it would determine the dependence of the chronos ψ_n with respect to the first one [as in (2.7)].

Finally, we would like to stress the fact that the indeterminacy in U described above in (i), in the case of a degenerate spectrum, can manifest itself as an internal bifurcation[3] through which a phase jump may occur when there is a crossing of eigenvalues and a redefinition of the spatio-temporal structures at a critical parameter value.

3. SCALING EXPONENT AND FOURIER SPECTRUM

In this section, we study the spatial two-point correlation function and its Fourier transform. In the K41 classical theory of fully developed turbulence, Kolmogorov deduced his universal $k^{-5/3}$ law of the Fourier spectrum from a scaling law of the spatial two-point correlations, assumed to be homogeneous. We now point out that our analysis is consistent with Kolmogorov's famous power law[16] (and Frisch's "modern" viewpoint[12]) by showing that the consideration of quasisymmetries, as introduced in the previous section, as well as the assumption that the scaling exponent is equal to 1/3 (which comes from physical considerations, namely that there is a finite and constant rate of energy dissipation per unit mass), leads to a decrease of the spatial correlations as $|x-y|^{-2/3}$ and thus to the $k^{-5/3}$ power law for the Fourier coefficients. Equivalently, if we assume the decreasing law of the correlations, the value of the scaling exponent can be deduced, and consequently the $k^{-5/3}$ power law spectrum.

It is interesting to note that the considerations of temporal averages necessary to realize a connection between the ensemble statistics and the temporal statistics leads us to choose the scalar products (1.4) and (1.5) instead of those defined in $L^2(R^3)$ and $L^2(R)$ used in the previous section. We now consider that the spatial and temporal domains K and T are fixed and we introduce the operator U from $L^2(K)$ to $L^2(T)$ whose kernel is $u(\mathbf{x}, t)$. The unitary representations of R_+ in $L^2(K)$ and $L^2(T)$ can be simply written as

$$(S_\lambda \varphi)(\mathbf{x}) = \varphi(\lambda^{-1}\mathbf{x}) \tag{3.1}$$

$$(\tilde{S}_\lambda \psi)(t) = \psi(\lambda^{-(1-h)}t) \tag{3.2}$$

In this case, the commutation relation becomes

$$\tilde{S}_\lambda U = \lambda^{-h} U S_\lambda \tag{3.3}$$

It directly follows that the spatial correlation operator U^*U and the temporal correlation operator UU^* satisfy the relations

$$S_\lambda^{-1} U^* U S_\lambda = \lambda^{2h} U^* U \tag{3.4}$$

$$\tilde{S}_\lambda^{-1} U U^* \tilde{S}_\lambda = \lambda^{2h} U U^* \tag{3.5}$$

Now, we use the fact that the kernel R_λ of the operator $S_\lambda^{-1}U^*US_\lambda$ is simply the rescaled kernel of U^*U:

$$R_\lambda(\mathbf{x}, \mathbf{y}) = R(\lambda\mathbf{x}, \lambda\mathbf{y}) \tag{3.6}$$

The assumption of the value of the scaling exponent $h = 1/3$ leads, in the homogeneous case, to the decay of the spatial correlation as $|\mathbf{x} - \mathbf{y}|^{-2/3}$ by combining (3.4) and (3.6). Equivalently, the assumption of the decay of the spatial correlation as $|\mathbf{x} - \mathbf{y}|^{-2/3}$ leads to the determination of $h = 1/3$. We stress here that although the homogeneity assumption needs to be made for the spatial correlation to depend on $|\mathbf{x} - \mathbf{y}|$ only, the derivation of (3.4) and (3.5) does not depend on this hypothesis (nor on the isotropy and the stationarity of the flow).

Now, by substituting the value $h = 1/3$ in (3.5) and denoting by $\tilde{R}_\lambda$ the kernel of the operator $\tilde{S}_\lambda^{-1}UU^*\tilde{S}_\lambda$, we get the following relation for the temporal correlation:

$$\tilde{R}_\lambda(t, s) = \tilde{R}(\lambda^{2/3}t, \lambda^{2/3}s) \tag{3.7}$$

Since (3.1) is valid independent of the dimension of $\mathbf{x}$, we now substitute in all the above formulas $\mathbf{x}$ by the longitudinal variable x (for comparison with the one-dimensional Fourier transform, as commonly used in turbulence). If we suppose that the spatial correlation is homogeneous in this direction, i.e.,

$$R(x, y) = R(|x - y|) = R(r) \tag{3.8}$$

and define its Fourier transform by

$$E(k) = \hat{R}(k) = \int_{-\infty}^{+\infty} e^{-ik\cdot r}R(r)\, dr \tag{3.9}$$

we then deduce from (3.4) and (3.6) that

$$E(\lambda k) = \hat{R}(\lambda k) = \int_{-\infty}^{+\infty} e^{-i\lambda k\cdot r}R(r)\, dr = \lambda^{-5/3}E(k) \tag{3.10}$$

If in addition the flow is assumed to be isotropic, $E(k)$ is related to the three-dimensional Fourier spectrum (integral in Fourier space of the trace of the spectrum tensor over a circular sphere of radius $|k|$) and both spectra exhibit the same power law. Note that if h is kept throughout the previous derivation, $E(k)$ decreases as $k^{-(2h+1)}$, which of course becomes $k^{-5/3}$ for the particular value $h = 1/3$. This power law in k may appear confusing since the kinetic spectrum (which coincides with the Fourier spectrum in homogeneous turbulence) decreases exponentially in n, according to (2.5). Both laws are, however, equivalent due to the fact that the wavelength of the topos φ_n, then identified with Fourier modes, decreases exponentially fast as a function of n. Here again the derivation of (3.10) does not depend on the particular form of the symmetry $\tilde{S}$ (assumed to be $\tilde{S}_\lambda$ above). It would be interesting to investigate whether the scaling (3.7) for the temporal two-point correlation is valid or not.

Finally, we end this section by the following remark, which we will return to in Section 5. The assumptions of homogeneity and stationarity do not play any role in the derivation of the scaling laws (3.4) and (3.5), except in the fact that we suppose that the result is independent of both the size and the localization of the window $K \times T$. Indeed, we should understand the symmetry (3.3), and therefore the scaling laws it leads to, in the limit where the integration domain $K \times T$ tends to infinity, when, of course, this limit exists. We should also mention that the symmetry (3.3) cannot be exact. The scaling relations between the spatial structures φ_n show that the A_n and thus the $E(k)$ do not satisfy the ideal scaling at both ends of the spectrum, but only in the inertial subrange (see Section 2).

4. THE KINETIC SPECTRUM

In this section, we study the spectrum of the operator U in the presence of various symmetries of the type (2.1). As shown in Section 2, the existence of a representation of Z which appears as a quasisymmetry (2.5) implies that the kinetic spectrum is discrete and follows an exponentially fast decreasing law. If we consider the scaling exponent h fixed, as in the previous section, then the scales of A_n, φ_n, and ψ_n depend only on λ and the specific value of h.

Now, let us suppose the presence of another symmetry (2.4) in the velocity field u defined with the same scaling exponent h but with a different scaling coefficient $\lambda' \neq \lambda$. What are then the consequences of this new symmetry on the kinetic spectrum? A simple analysis of this situation shows that two and only two cases are possible:

(i) The ratio $\ln \lambda / \ln \lambda'$ is a rational number, namely $\ln \lambda / \ln \lambda' = p/q$, $p \in N$, $q \in N$. In this case, the spectrum is still discrete and exponentially decreasing with n, with a dilation factor $\lambda'' = \lambda^{1/q} = (\lambda')^{1/p}$.

(ii) Otherwise, the kinetic spectrum coincides with R_+ because then the closure of the set $\{\lambda^n (\lambda')^m, n, m \in Z\}$ is R_+.

In view of the preceding analysis, we can conjecture a possible route toward turbulence in three stages, as described by the kinetic spectrum. We emphasize here that this scenario depends only on the turbulence (kinetic) spectrum and therefore ignores the particular form of the velocity field $u(\mathbf{x}, t)$ as a function of space and time. Two totally different flows can thus follow the same route, provided their spectra enjoy the same property as described below. A simple example of this situation is given by two flows which differ by their symmetries $\tilde{S}$ (see our remark above) or by symmetries other than scaling symmetries (since the multiplicity does not affect the spectrum). In the following, we assume that a scaling symmetry in the fluid flow can appear at a finite (instead of infinite) Reynolds number (in the inertial range) as has been indeed observed in experiments. The first stage would be a traveling wave (usually present after the first instability in open flows) which corresponds to the degeneracy of the kinetic spectrum. A traveling wave, which is due to the existence of a temporal and spatial translation symmetry,[4] can be interpreted as a symmetry of the form (2.5) with a scaling factor $\lambda = 1$ and translation coefficients. Then, as the Reynolds number, or another relevant bifurcation parameter, say ε, varies, the deviation of the scaling factor from the value 1 would lead to an exponentially decreasing discrete spectrum with a unique value $\lambda(\varepsilon)$. A special case of this scenario could consist in the appearance of a symmetry with scaling factor λ in the traveling wave, in which case the degenerate spectrum would become exponentially decreasing, as, for instance, in the case of the degenerate traveling wave train described in ref. 4. To this regime, where only one scaling property with $\lambda \neq 1$ is present, would immediately succeed the complex regime characterized by the fact that the kinetic spectrum described above becomes R_+ as soon as another symmetry of scaling factor $\lambda' \neq 1$, logarithmically incommensurable with the first one, appears. We should keep in mind that the variation of λ in the intermediate phase, as a function of the bifurcation parameter ε, as well as the appearance of a second λ', are probably governed by conditions on the evolution of the global energy or entropy[3] which the spectrum can absorb. For example, the latter (per unit volume and time) is obviously limited by P_Δ and $\tilde{P}_\Delta$, that is, by the boundary conditions, which control the energy-containing range, and the Reynolds number, which determines the dissipation range. It is not clear why, at a certain parameter value, a new scaling symmetry appears (which is enough to provide a spectrum of U identified with R_+, the maximum spectrum for U, i.e., a spectrum without gaps). In the intermediate phase, the kinetic spectrum, although discrete, could consist of several families of eigenvalues with various decay rates. This could arise as a superposition of symmetries as described in ref. 4, because then the spectrum of U is only the union of the spectra of the various restrictions of U to the eigenspaces on which the symmetries act. But, in order for the spectrum to be maximal, namely R_+, a second symmetry, logarithmically incommensurable with the first one, has to appear at least in one of these eigenspaces.

If we suppose that the kinetic spectrum is absolutely continuous in fully developed turbulence, the spectral density of the spatial correlation can be written as (see Appendix A)

$$E^*(A) = A^2 f(A) \tag{4.1}$$

where f satisfies the scaling property (A.36). If the flow satisfies the scaling property [(2.9), (2.10)], we obtain a scaling law which generalizes (3.10):

$$E^*(\lambda^h A) = \lambda^h E^*(A) \tag{4.2}$$

If the flow is invariant under a full symmetry, (2.9), (2.10) are satisfied for all positive λ, then we obtain from (4.1) and (4.2)—as well as from (A.36)—a decay law for f:

$$f(A) \propto A^{-1} \tag{4.3}$$

namely $E^*(A)$ is simply proportional to A. Finally, taking

$$f(A)\, dA = \sum_n \delta(A - A_n) \tag{4.4}$$

in order to recover the case of a discrete spectrum, we get the particular form of (4.3) in this case:

$$A_{n+1}^2 = \lambda^{2h} A_n^2 \qquad \text{for all } n \tag{4.5}$$

Since, if the flow is homogeneous, the spectral density (4.2), or (4.5) in the discrete case, coincides with that of the Fourier spectrum (see above), the universal law K41 is satisfied (for $h = 1/3$). In view of the remark in Section 2, if the symmetry holds only in a "statistical" sense [namely (2.9) only is satisfied], then the spectrum laws (4.3) and (4.5) are still valid.

Finally, we will show how the above scenario for the kinetic spectrum can become even more complex than in the case treated above. This occurs when the velocity has symmetries of the type (2.1) localized in space and time. Such a situation is treated in the next section.

5. SPACE-TIME LOCAL STRUCTURES

Experimental and numerical studies have shown that the global self-similarity of turbulence is not exact and that the flow is intermittent (see, e.g., Kuo and Corrsin,[18] She *et al.*,[23] Castaing *et al.*,[7] and Vincent and Meneguzzi[26]). Pushing further the ideas of the multifractal model of Parisi and Frisch,[22] it is conceivable that the symmetries and singularities of the velocity field are nonuniformly distributed in space and time and that they are responsible for the intermittency effects and the behavior of the higher moments of the statistics.

So far, we have treated the case where the velocity field $u(\mathbf{x}, t)$ satisfies the (global) symmetry property (2.0), that is, $u(\mathbf{x}, t)$ has a local symmetry only at the point $(\mathbf{x}, t) = (\mathbf{0}, 0)$. However, following the previous arguments, it is most probable that there are various space-time locations $(\mathbf{x}_0, t_0)$ at which the velocity exhibits a particular dynamics, eventually a singularity of a certain order. This leads to modifying equation (2.1) and introducing multiple scaling exponents as allowed by the invariance of the Navier–Stokes equations in the limit of zero viscosity given by (2.1). A theory which takes into account singularities (in space) of various orders is that of multifractals, as introduced in turbulence by Parisi and Frisch[22] (see also ref. 12 for a review on the subject).

In our context, this necessary generalization of (2.1) becomes very natural when the description has to adjust itself to the window $K \times T$ defining the integration domain. We now show how such a situation can be easily introduced in our framework and we also what the consequences are for the spectral properties of U. We stress that it is very important to first find a spatiotemporal version of the symmetry. We take it under the following form, which is the direct generalization of (2.1) when the symmetry is not restricted to a unique dilation center $(\mathbf{x}_0, \mathbf{t}_0)$.

Local Symmetry Property (LS). There exists a function $h\colon R^3 \to R$ and a function $\tilde{h}\colon R \to R$ such that

$$u(x - x_0, \alpha(t_0)(t + t_0)) = \beta(x_0)\, u(\lambda^{-1}x - x_0, t - t_0) \tag{5.1}$$

for all (x_0, t_0) such that

$$h(x_0) = \tilde{h}(t_0) \tag{5.2}$$

where we have used the notation

$$\alpha(t_0) = \lambda^{1 - \tilde{h}(t_0)} \qquad \text{and} \qquad \beta(x_0) = \lambda^{h(x_0)} \tag{5.3}$$

and fixed λ for the moment. The points (x_0, t_0) which satisfy this property are the dilation centers. Remark that the condition

$$h(x_0) = \tilde{h}(t_0) \tag{5.4}$$

comes from (and is equivalent to) the relation

$$\alpha(t_0)\, \beta(x_0) = \lambda \tag{5.5}$$

It then suffices to apply the same technique as in Section 2 to derive the commutation relations needed in our analysis. For this, we have to consider the subgroups of spatial and temporal dilations–translations defined with the powers of the operators:

$$(S_{\lambda, x_0}\varphi)(x) = \lambda^{-3/2}\varphi(\lambda^{-1}(x + x_0) - x_0) \tag{5.6}$$

and

$$(\tilde{S}_{\alpha(t_0), t_0}\psi)(t) = \alpha(t_0)^{-1/2}\, \psi(\alpha(t_0)^{-1}\, (t + t_0) - t_0) \tag{5.7}$$

We then derive the following commutation relation, which is equivalent to the LS property:

$$US_{\lambda, x_0} = \lambda^{h(x_0)/2 + 2}\tilde{S}_{\alpha(t_0), t_0} U \tag{5.8}$$

From this relation, we deduce that the kinetic spectrum consists of an exponentially decreasing family of eigenvalues if the different values of the function h which appear in (5.8) are rationally related, leading to a degeneracy corresponding to

$$UT_{x_1 - x_0} = \tilde{T}_{t_1 - t_0} U \tag{5.9}$$

where T_ξ denotes the translation operator:

$$(T_\xi \varphi)(x) = \varphi(x - \xi) \tag{5.10}$$

and similarly $\tilde{T}_\tau$ is defined as

$$(\tilde{T}_\tau \psi)(t) = \psi(t - \tau) \tag{5.11}$$

On the contrary, if $h(x_0)/h(x_1)$ is irrational for two different dilation centers (x_0, t_0) and (x_1, t_1), the kinetic spectrum becomes all R_+. Note again that all this is true if the symmetry (5.8) holds only in a statistical sense, namely we only know that scaling is valid for the two spatial point correlations only, i.e.,

$$S_{\lambda, x_0}^{-1} U^* US_{\lambda, x_0} = \lambda^{h(x_0) + 4} U^* U \tag{5.12}$$

This situation can appear at first similar to that met in the previous section when the kinetic spectrum becomes R_+. We could, however, obtain a discrete spectrum dense in R_+ in the present case, as a consequence of the spreading of the dilation centers, which may lead to possible singularities. The dynamics should be different in this case compared to that observed in the situation described in Section 4, where only one such point exists. Note that there the symmetries should be considered in a local sense since, as we go away from the dilation center (x_0, t_0) (or perhaps a hypersurface of such centers), the velocity u has to be matched to the next symmetry with (x_1, t_1) as the new center. Thus, in the matching region, (5.6) is not exactly satisfied. Then, a simple perturbation argument leads to the conclusion that local deformations or defaults appear in the topos and chronos in these regions. [This deformation should be understood with respect to the corresponding topos and chronos for the families satisfying the global symmetry (2.6), (2.7).] It is clear that, as in Section 2, the derivation is possible in the spaces $H(K)$ and $H(T)$ corresponding to a "window" leading to identical formulas with the appropriate exponents, namely, (5.8) and (5.12) should be replaced by

$$US_{\lambda, x_0} = \lambda^h \tilde{S}_{\alpha(t_0), t_0} U \tag{5.13}$$

$$S^{-1}_{\lambda, x_0} U^* U_{\lambda, x_0} = \lambda^{2h(x_0)} U^* U \tag{5.14}$$

Finally, we remark that, as in homogeneous, isotropic turbulence, the definition of the inertial range at finite (but large) Reynolds number is more complex than in (K41) since the dissipative range should now depend on the values of h present in the local scaling: in our formalism, there should be two projectors $\tilde{P}_\Delta$ and P_Δ, for each value of the scaling exponent h.

6. CONCLUDING REMARKS

Extending the technique used by Kolmogorov to derive the $k^{-5/3}$ power law of the energy spectrum in isotropic, homogeneous turbulence, we have derived scaling laws for the turbulence spectrum of inhomogeneous turbulence in the inertial range, namely where scaling occurs. However, no assumption has been made regarding the transfer of energy down the spectrum, so that the value of h is not determined (the value $h = 1/3$ is still probably approximately valid). If the flow is invariant only under one subgroup of the scaling coefficient λ of the group of dilations, the spectrum is discrete and follows an exponential law of decay rate λ^{2h}, namely $A^2_{n+1} = \lambda^{2h} A^2_n$ for all n. If the flow is invariant under two subgroups of scaling coefficients λ and λ', then the spectrum becomes R_+. If the flow is invariant under the whole group of dilations $\lambda > 0$ and the spectrum is continuous, the spectral density of the spectrum $E^*(A)$ is found to be independent of h and equal to A. An interesting point is that these spectrum laws do not depend on whether the scaling of the flow is deterministic or statistical, that is, whether it applies to the velocity (and consequently to the spatial and temporal two-point correlations) or to the spatial two-point correlation only. In both cases, two symmetries, one spatial and one temporal, are present in the flow, the main difference being that in the first case, the temporal symmetry acts on the time variable (and is of the type present in the Navier–Stokes equations, namely a scaling transformation), while in the second case, it may be more complex. It would be interesting to investigate the nature of this symmetry from experimental or numerical data. However, the fact that in any case both symmetries exist should have important consequences for turbulence modeling and numerical simulations. It simply means that, provided that these symmetries are known, one needs to find one spatiotemporal structure (ψ_0, φ_0) only, all the other structures being systematically deduced. Since our results are consequences of scaling laws alone, they are valid in the inertial range only, so that a model or calculation based on the previous remarks would need to input both the energy-containing and the dissipative range. Although it is common to model the dissipative range in a rather crude way (as in "large-eddy simulations"), it may be possible to apply the ideas developed in this paper to this range as well, as indicated in the work of Foias *et al.*,[11] who showed, based on the Gevray class regularity of the Navier–Stokes equations, that the spectrum should decay exponentially fast in the dissipative range as well. However, symmetries may be different there from those valid in the inertial range. It is interesting to note that Knight and Sirovich[15] point out the existence of an inertial range in various numerical simulations of inhomogeneous turbulent flows. However, the scaling law they propose, namely $A_n^2 \propto n^{-11/9}$, based on a heuristic argument, is fundamentally different from ours. Finally, in the same way in which modifications have been made to explain deviations from the K41 law by the introduction of the multifractal concept,[22] we were also able to define, in our case, local symmetries in space and time and deduce consequences for the spectrum.

Finally, we would like to emphasize that all techniques used in this paper for the Navier–Stokes equations and their "chaotic" (i.e., turbulent) solutions are applicable to any other partial differential equations and their "self-similar" solutions which are invariant under scaling (or even more complex) transformations. If the scaling invariance is valid in one direction only (meaning in either space or time), then fractal and multifractal properties are recovered by our technique. The latter, however, is not limited to these "one-dimensional" cases and can be applied in the same manner if the scaling transformation involves both space and time, as we have shown in this paper. Temporal and spatial fractals/multifractals are then recovered for the temporal and spatial two-point correlations (whose properties, such as the fractal dimension, may be different).

APPENDIX A

In this technical appendix, we generalize biorthogonal decompositions, beyond the case treated in ref. 3, where the operator U, whose kernel is $u(\mathbf{x}, t)$, is bounded and compact. This generalization, made necessary by the present work, includes the case where the spectrum of U has a continuous component as well as the case where U is an unbounded operator with eventually a singular kernel (Carleman kernel operator) in order to include the eventual divergences of u.

It is our point of view that the best way to treat all these cases at once is to use the powerful theory of eigenfunction expansion of self-adjoint operators. This theory, which goes back to Krein,[17] has been developed by many others. It has been largely completed by Berezansky,[6] and the interested reader can find abundant information and bibliography in ref. 6. In our framework, such a theory has the advantage of permiting the treatment of cases much more general than that of bounded and compact operators with a very similar formalism as that used in the simpler case.[3,4] In brief, the method exposed in ref. 6 permits us to write the decomposition

$$\mathbf{u}(\mathbf{x}, t) = \sum_{n=0}^{N} A_n \varphi_n(\mathbf{x}) \psi_n(t) \tag{A.1}$$

(N being either finite or infinite) when the spectrum is discrete if the functions φ_n (resp. the functions ψ_n) are taken in a space "slightly larger" than the space $L^2(R^3)$ [resp. $L^2(R)$] on which the operators U and U^* act. If the spectrum is continuous, a formula similar to (A.1) holds where the sum must be replaced by an integral [see below, (A.14)]. Essentially, we adapt here Theorem V-4-1 of ref. 6 to the case where the operator acts from one space to the other, which is the essential step to realize the biorthogonal decomposition. For a better understanding, we now recall some basic notions from ref. 6. One has to consider chains of three separable Hilbert spaces

$$H_- \supseteq H_0 \supseteq H_+ \tag{A.2}$$

whose respective norms satisfy

$$\|u\|_- \leqslant \|u\|_0 \leqslant \|u\|_+ \tag{A.3}$$

and such that H_+ is a dense subspace of H_0 for the norm $\|\cdot\|_0$ and similarly H_0 is a dense subspace of H_- for the norm $\|\cdot\|_-$. Moreover, the quasinuclearity of the embedding H_- is needed below, namely the operator J defined by

$$(Jf, Jg)_+ = (f, g)_0, \qquad f, g \in H_0 \tag{A.4}$$

considered as an operator on H_0 has to be Hilbert–Schmidt. Very often, it is possible to associate a sequence of Hilbert spaces as defined in (A.2) satisfying the above conditions to a self-adjoint operator A acting on H_0 with a dense domain. Then, the technique consists in using H_- as the space of "generalized eigenvectors" of A.

A particular realization of such a chain is given by

$$L^2\left(Q, \frac{1}{p}\,dq\right) \supseteq L^2(Q, dq) \supseteq L^2(Q, p\,dq) \tag{A.5}$$

where Q is a separable locally compact space, dq a Borel measure (in this paper, $Q = R^3$ or R and $dq = d\mathbf{x}$ or dt), and p a measurable function with $p(q) \geqslant 1$. Here, the space $L^2(Q, p\,dq)$ is defined by the scalar product:

$$(\xi, \eta)_+ = \int_Q \xi(q)\,\overline{\eta(q)}\,p(q)\,dq < +\infty \tag{A.6}$$

and similarly the space $L^2(Q, (1/p)\,dq)$ is defined by

$$(\alpha, \beta)_- = \int_Q \alpha(q)\,\overline{\beta(q)}\,\frac{1}{p(q)}\,dq < +\infty \tag{A.7}$$

Then, we can write the following (from Theorem V-4-1 in ref. 6 adapted to the case of the operator U). Let us suppose that for the operator U from $L^2(R^3, dx)$ to $L^2(R, dt)$, corresponding to the kernel $u(\mathbf{x}, t)$ by (1.1) and (1.2), the operators U^*U and UU^* satisfy the conditions of Theorem V-4-1 of ref. 6, namely:

(1) There is a positive, bounded, and continuous function γ defined on R_+ such that the operator $\gamma(U^*U)$ is an integral operator acting on $L^2(R^3, d\mathbf{x})$ whose kernel $R_\gamma(\mathbf{x}, \mathbf{y})$ satisfies

$$\int |R_\gamma(\mathbf{x}, \mathbf{y})|^2\,d\mathbf{x} < +\infty \tag{A.8}$$

for almost all $\mathbf{y} \in R^3$ (with respect to $d\mathbf{x}$), and $\gamma(UU^*)$ is an integral operator acting on $L^2(R, dt)$ whose kernel satisfies

$$\int |\tilde{R}_\gamma(s, t)|^2\,ds < +\infty \tag{A.9}$$

for almost all t (with respect to dt), and

(2) There exists a measurable function $p_X(\mathbf{x}) \geqslant 1$, $\mathbf{x} \in R^3$, such that

$$\iint |R_\gamma(\mathbf{x}, \mathbf{y})|\,\frac{1}{p_x(\mathbf{y})}\,d\mathbf{x}\,d\mathbf{y} < +\infty \tag{A.10}$$

and a measurable function $p_T(t) \geqslant 1$, $t \in R$, such that

$$\iint |\tilde{R}_\gamma(s, t)| \frac{1}{p_T(t)} ds\, dt < +\infty \tag{A.11}$$

Then, we can write the decomposition of $u(\mathbf{x}, t)$ as

$$u(\mathbf{x}, t) = \int_0^{+\infty} A \sum_{n=1}^{N_A} \varphi_n^A(\mathbf{x}) \psi_n^A(t)\, dm(A) \tag{A.12}$$

where the measure dm is finite and has support on the spectrum of U, N_A is finite or infinite, and the series $\varphi_n^A \in L^2(R^3, (1/p_X)\, d\mathbf{x})$ and $\psi_n^A \in L^2(R, (1/p_T)\, dt)$ converge in the metric of the tensorial product of the two spaces. The decomposition (A.12) holds for almost all $\mathbf{x} \in R^3$ and $t \in R$.

In order to reduce our case to Theorem V-4-1 of ref. 6, we consider the following chain [see (A.2)]:

$$H_- \supseteq H_0 \supseteq H_+ \tag{A.13}$$

where

$$H_0 = L^2(R^3, d\mathbf{x}) \oplus L^2(R, dt) \tag{A.14}$$

$$H_+ = L^2(R^3, p_X\, d\mathbf{x}) \oplus L^2(R, p_T\, dt) \tag{A.15}$$

$$H_- = L^2\left(R^3, \frac{1}{p_X} d\mathbf{x}\right) \oplus L^2\left(R, \frac{1}{p_T} dt\right) \tag{A.16}$$

If we now define a set of 2×2 matrices e_{ij}, $j = 1, 2$, such that

$$e_{ij} e_{kl} = \delta_{jk} e_{il} \tag{A.17}$$

then the operator

$$J = J_X \otimes e_{11} + J_T \otimes e_{22} \tag{A.18}$$

where J_X (resp. J_T) is the embedding (A.4) for the spatial (resp. temporal) chain (A.5) with $Q = R^3$ (resp. $Q = R$), is Hilbert–Schmidt and therefore (A.13) is quasinuclear.

If we then define the operator V as

$$V = U^* \otimes e_{12} + U \otimes e_{21} \tag{A.19}$$

which is a self-adjoint operator on H_0 satisfying

$$V^2 = U^*U \otimes e_{11} + UU^* \otimes e_{22} \tag{A.20}$$

and take $\tilde{\gamma}(A)=\gamma(A^2)$, from Theorem V-4-1 of ref. 6, we deduce the decomposition of V in generalized vectors defined in H_- as well as the fact that $D(V)\cap H_+$ [$D(V)$ denotes the domain of V as in (1.2)] is dense in H_0, so that we can write

$$\forall \xi, \eta \in D(V)\cap H_+, \qquad (V\xi, \eta)=\int A \sum_{n=1}^{N_A} (\xi, \phi_n^A)(\phi_n^A, \eta)\, dm(A) \qquad \text{(A.21)}$$

Therefore, by writing $\phi_n^A=\varphi_n^A\oplus\psi_n^A$ with $\varphi_n^A\in L^2(R^3, (1/p_X)\, d\mathbf{x})$ and $\psi_n^A\in L^2(R, (1/p_T)\, dt)$ and taking $\xi=\xi_X\oplus 0$ and $\eta=0\oplus\eta_T$, we obtain the relation

$$(U\xi_X, \eta_T)=\int A \sum_{n=1}^{N_A} (\xi_X, \varphi_n^A)(\psi_n^A, \eta_T)\, dm(A) \qquad \text{(A.22)}$$

Since H_+ is dense in H_0, (A.12) follows by identification of the kernels, for m-almost all A. We call the functions φ_n^A generalized topos and the functions ψ_n^A generalized chronos. Notice that even in this case the exact dispersion relation (the one-to-one correspondence between topos and chronos) is preserved, of course for m-almost all A.

Remark. On the one hand, since R_γ is the kernel of $\gamma(U^*U)$ and not that of U^*U, and similarly $\tilde{R}_\gamma$ is the kernel of $\gamma(UU^*)$ and not that of UU^*, the conditions (A.10) and (A.12) allow us to treat the case of a non-bounded operator U. Most of the time, γ is the resolvant function of some positive power of $|u|$:

$$\gamma(A)=\frac{1}{(A^k-z)}, \qquad \mathrm{Im}(z)\neq 0 \qquad \text{(A.23)}$$

On the other hand, the weights p_X and p_T allow the treatment of kernels with singularities. The symmetries are introduced in a similar way as in ref. 4, except that we now need to define them as unitary operators acting on the largest spaces, namely

$$S_\lambda\colon\ L^2\left(R^3, \frac{1}{p_X} d\mathbf{x}\right)\to L^2\left(R^3, \frac{1}{p_X} d\mathbf{x}\right) \qquad \text{(A.24)}$$

and

$$\tilde{S}_\lambda\colon\ L^2\left(R, \frac{1}{p_T} dt\right)\to L^2\left(R, \frac{1}{p_T} dt\right) \qquad \text{(A.25)}$$

It then suffices to take the images of the corresponding symmetries in $L^2(R^3, dx)$ and $L^2(R, dt)$. For instance, suppose that the kernels are polynomially singular. Then, we take

$$p_X(\mathbf{x}) = |\mathbf{x}|^{2a} \tag{A.26}$$

$$p_T(t) = |t|^{2b} \tag{A.27}$$

and define the symmetry operators by

$$(S_\lambda \varphi)(\mathbf{x}) = \lambda^{(a-3/2)} \varphi(\lambda^{-1}\mathbf{x}) \tag{A.28}$$

$$(\tilde{S}_\lambda \psi)(t) = \alpha^{(b-1/2)} \psi(\alpha^{-1}t) \tag{A.29}$$

If we now use the invariance of u (for a given λ and h), i.e.,

$$u(x, t) = \beta u(\lambda^{-1}x, \alpha^{-1}t) \tag{A.30}$$

with $\alpha\beta = \lambda$ and $\alpha = \lambda^{1-h}$, $\beta = \lambda^h$, we get from (A.22) and (A.12)

$$\begin{aligned} \beta u(\lambda^{-1}\mathbf{x}, \alpha^{-1}t) &= \beta \int_0^{+\infty} A \sum_{n=1}^{N_A} \varphi_n^A(\lambda^{-1}\mathbf{x})\, \psi_n^A(\alpha^{-1}t)\, dm(A) \\ &= \tilde{\beta}^{-1} \int_0^{+\infty} A \sum_{n=1}^{N_A} S_\lambda \varphi_n^A(\mathbf{x})\, \tilde{S}_\lambda \psi_n^A(t)\, dm(A) \\ &= \int_0^{+\infty} B \sum_{n=1}^{N_A} S_\lambda \varphi_n^{\tilde{\beta}B}(\mathbf{x})\, \tilde{S}_\lambda \psi_n^{\tilde{\beta}B}(t)\, dm(A) \end{aligned} \tag{A.31}$$

where we have used the notation $B = \tilde{\beta}^{-1}A$ and $\tilde{\beta}^{-1} = \beta\lambda^\rho\alpha^\tau$, the powers ρ and τ depending only on h, p_X, and p_T. For instance, with p_X and p_T defined as in (A.26) and (A.27), we have

$$\rho = 3/2 - a \tag{A.32}$$

$$\tau = 1/2 - b \tag{A.33}$$

Clearly, a and b are needed in order to eliminate a possible singularity due to h and thus the exponent of λ appearing in $\tilde{\beta}^{-1}$, that is, $h + \rho + (1-h)\tau$, only depends on h. Now, by identifying (A.12) and (A.31), we can conclude as follows:

(P1) The spectrum of U, sp(U), is invariant by scaling

$$\text{sp}(U) = \tilde{\beta}^{-1}\, \text{sp}(U) \tag{A.34}$$

with the multiplicity $N_B = N_{\tilde{\beta}^{-1}B}$, N_B being finite or infinite.

(P2) For the absolute continuous part of the measure, let f be the spectral density

$$dm_c(A) = f(A)\, dA$$

Since

$$dm_c(B) = \frac{\tilde{\beta}^{-1} f(\tilde{\beta}^{-1} A)}{f(A)}\, dm_c(A) \tag{A.35}$$

we get from (A.31) the scaling of this density:

$$\tilde{\beta}^{-1} f(\tilde{\beta}^{-1} A) = f(A) \tag{A.36}$$

for almost all B with respect to this measure.

(P3) P2 is true also for the singular part of the measure if we take the density of each component with respect to an arbitrary measure in the same class.

(P4) As before, we get

$$\varphi_B^n = S_\lambda \varphi_{\tilde{\beta}B}^n, \qquad n = 1,..., N_B \tag{A.37}$$

and

$$\psi_B^n = \tilde{S}_\lambda \psi_{\tilde{\beta}B}^n, \qquad n = 1,..., N_B \tag{A.38}$$

Therefore, we have the exact generalization of the symmetries introduced in ref. 4. For this reason, we use the same notation and write

$$\tilde{S}_\lambda U = \tilde{\beta} U S_\lambda$$

which has the meaning of Propositions P1–P4 stated above. Note that the scaling $\tilde{\beta}$ only depends on h, for a given $\lambda \in R^+$.

As before, it is clear that the same formulas are valid in the case of an operator defined in the "window" $K \times T$ with again the obvious change in the definition of $\tilde{\beta}$, which is then λ^{-h} [see (2.8)].

APPENDIX B

The reader familiar with the well-known wavelet analysis has probably noticed that, in the presence of symmetries of the type (2.2), (2.3) or (5.6), (5.7), the biorthogonal decomposition of $u(\mathbf{x}, t)$ (in the discrete case)

$$u(\mathbf{x}, t) = \sum_n A_n \varphi_0(\lambda^{-n}\mathbf{x} - \mathbf{x}_n)\, \psi_0(\alpha^{-n} t - t_n) \tag{B.1}$$

with

$$\mathbf{x}_n = \frac{\lambda^{-(n-1)} - 1}{1 - \lambda} \mathbf{x}_0, \qquad t_n = \frac{\alpha^{-(n-1)} - 1}{1 - \alpha} t_0 \tag{B.2}$$

and

$$A_n = \gamma^n A_0 \tag{B.3}$$

coincides with a spatiotemporal wavelet decomposition which automatically selects spatial and temporal wavelet mothers φ_0 and ψ_0, the spatial and temporal dilation factors λ and α, and therefore the sequences of spatial and temporal translations x_n and t_n. Even if the velocity field does not perfectly satisfy such a symmetry, the systematic selection of wavelet mothers in the inertial range (to avoid boundary conditions and viscous effects) could appear as a judicious choice. Following the hat fashion introduced in the literature for the classical wavelets, these wavelet mothers, necessarily adapted to each velocity field considered, could be called "barretes."[4] Here again, we see how important it is, from our point of view, to consider spatial as well as temporal symmetries.

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[4] From the name of a Portuguese hat which adjusts itself to the head it covers.

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